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Convergence of Cascade Algorithms in Sobolev Spaces for Perturbed Refinement Masks

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In this paper, the convergence of the cascade algorithm in a Sobolev space is considered if the refinement mask is perturbed. It is proved that the cascade algorithm corresponding to a slightly perturbed mask converges. Moreover, the perturbation of the resulting limit function is estimated in terms of that of the masks. © 2002 Elsevier Science (USA)

Key Words: cascade algorithm; Sobolev space; joint spectral radius; perturbation of refinable functions.

1. INTRODUCTION

In this paper, we are concerned with the following problem: Given a compactly supported multivariate refinable function ϕ, how does perturbation of its finite refinement mask affect the convergence of the cascade algorithm? Further, if the cascade algorithm for the perturbed mask also converges, how the resulting limit function is related with ϕ?

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We say that a compactly supported function ϕ is M-refinable if it satisfies a refinement equation

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(M \cdot - \alpha), \tag{1.1}$$

where the finitely supported sequence $a = (a(\alpha))_{\alpha \in \mathbb{Z}^s}$ is called the *refinement mask*. The $s \times s$ matrix M is called a *dilation matrix*. We suppose that its entries are integers and that $\lim_{k \to \infty} M^{-k} = 0$. Throughout the paper, we assume that M is *isotropic*. This means that there is an invertible matrix Λ such that

$$\Lambda M \Lambda^{-1} = \mathrm{diag}(\sigma_1, \ldots, \sigma_s)$$

with $|\sigma_1| = \cdots = |\sigma_s| = m^{1/s} = \varrho(M)$, where $m := |\det M|$ and $\varrho(M)$ is the spectral radius of M.

Let the Fourier transform $\hat{f}$ of a function $f \in L_1(\mathbb{R}^s)$ be defined by

$$\hat{f}(\omega) := \int_{\mathbb{R}^s} f(x) e^{-ix \cdot \omega}\, dx, \qquad \omega \in \mathbb{R}^s,$$

where $x \cdot \omega$ denotes the inner product of two vectors x and ω in $\mathbb{R}^s$. The Fourier transform is naturally extended to the space of all compactly supported distributions. We can rewrite Eq. (1.1) as

$$\hat{\phi}(M^T \omega) = H_a(\omega)\hat{\phi}(\omega), \qquad \omega \in \mathbb{R}^s, \tag{1.2}$$

where the refinement mask symbol

$$H_a(\omega) = \frac{1}{m} \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) e^{-i\omega \cdot \alpha}, \qquad \omega \in \mathbb{R}^s$$

is a (multivariate) trigonometric polynomial.

Looking at the refinement equation (1.1) as a functional equation, one can give necessary and sufficient conditions for the mask a to ensure existence, uniqueness and regularity of the solution ϕ_a (see e.g. [1] for $M = 2I$). Provided that

$$\sum_{\alpha \in \mathbb{Z}^s} a(\alpha) = m, \tag{1.3}$$

there exists a unique compactly supported distribution ϕ_a with $\hat{\phi}_a(0) = 1$ satisfying (1.1) (see e.g. [1, 22]). Throughout the paper, we assume that condition (1.3) holds for the refinement masks considered.

Before posing the problem more explicitly, we need to review some notations. For $1 \leqslant p \leqslant \infty$, the norm of $L_p(\mathbb{R}^s)$ is denoted by $\|\cdot\|_p$. Let

$$W_p(\mathbb{R}^s) := \begin{cases} L_p(\mathbb{R}^s), & 1 \leqslant p < \infty, \\ C_u(\mathbb{R}^s), & p = \infty, \end{cases}$$

where $C_u(\mathbb{R}^s)$ is the space of uniformly continuous and bounded functions on $\mathbb{R}^s$ equipped with norm $\|\cdot\|_\infty$. Further, we use the convention $1/\infty = 0$.

Let $\mathbb{Z}_+$ be the set of nonnegative integers and

$$\mathbb{Z}_+^s := \{(\mu_1, \ldots, \mu_s) \in \mathbb{Z}^s : \mu_i \geqslant 0 \quad \forall i = 1, \ldots, s\}.$$

For any multi-integer $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{Z}_+^s$, let $|\mu| := \mu_1 + \cdots + \mu_s$, $\mu! := \mu_1! \cdots \mu_s!$ and $x^\mu := x_1^{\mu_1} \cdots x_s^{\mu_s}$. Further, Π_n denotes the linear span of $\{x^\mu : |\mu| \leqslant n\}$. For two multi-integers $\mu = (\mu_1, \ldots, \mu_s)$ and $\nu = (\nu_1, \ldots, \nu_s)$, we say $\nu \leqslant \mu$ if $\nu_i \leqslant \mu_i$ for all $i = 1, \ldots, s$. For $\nu \leqslant \mu$, we use $\binom{\mu}{\nu}$ to denote $\frac{\mu!}{(\mu-\nu)!\nu!}$.

For $n \in \mathbb{Z}_+$, the Sobolev space $W_p^n(\mathbb{R}^s)$ is the set of all tempered distributions f such that $D^\mu f \in W_p(\mathbb{R}^s)$ for $|\mu| \leqslant n$, where $D^\mu = D_1^{\mu_1} \ldots D_s^{\mu_s}$ and $D_j := \frac{\partial}{\partial x_j}$ $(j = 1, \ldots, s)$ denote the partial derivatives. Clearly, $W_p^n(\mathbb{R}^s)$ is a Banach space with the norm

$$\|f\|_{W_p^n(\mathbb{R}^s)} := \sum_{|\mu| \leqslant n} \|D^\mu f\|_p, \qquad 1 \leqslant p \leqslant \infty.$$

Let E be a complete set of representatives of distinct cosets of the quotient group $\mathbb{Z}^s / M\mathbb{Z}^s$. Thus, each element $\alpha \in \mathbb{Z}^s$ can be uniquely represented as $\alpha = \varepsilon + M\gamma$, $\varepsilon \in E$ and $\gamma \in \mathbb{Z}^s$. It is known that the cardinality of E is equal to $m = |\det M|$. Without loss of generality, we can assume that $0 \in E$.

Denote by $\ell(\mathbb{Z}^s)$ the space of all complex-valued sequences. Let $\ell_p(\mathbb{Z}^s)$ be the space of complex-valued sequences $\lambda = (\lambda(\alpha))_{\alpha \in \mathbb{Z}^s}$ such that $\|\lambda\|_p < \infty$, where

$$\|\lambda\|_p := \begin{cases} \left(\sum_{\alpha \in \mathbb{Z}^s} |\lambda(\alpha)|^p \right)^{1/p}, & 1 \leqslant p < \infty, \\ \sup_{\alpha \in \mathbb{Z}^s} |\lambda(\alpha)|, & p = \infty. \end{cases}$$

(Observe that the norms for $W_p(\mathbb{R}^s)$ and $\ell_p(\mathbb{Z}^s)$ both are abbreviated by $\|\cdot\|_p$, the particular interpretation will always follow from the context.)

Denote by $\ell_0(\mathbb{Z}^s)$ the space of sequences of finite support. For $\lambda \in l_0(\mathbb{Z}^s)$ let $\operatorname{supp} \lambda := \{\alpha \in \mathbb{Z}^s : \lambda(\alpha) \neq 0\}$.

Given a compactly supported initial function $\phi_0 \in L_p(\mathbb{R}^s)$, we define a sequence $(\phi_k)_{k\geqslant 1}$ by iteration $\phi_k := Q_a\phi_{k-1}$, $k = 1, 2, \ldots,$ where $Q_a : L_p(\mathbb{R}^s) \mapsto L_p(\mathbb{R}^s)$ is the *cascade operator associated with the finite mask a*,

$$Q_a f := \sum_{\beta\in\mathbb{Z}^s} a(\beta) f(M\cdot -\beta). \tag{1.4}$$

We say that the cascade algorithm converges for ϕ_0 in $W_p^n(\mathbb{R}^s)$-norm $(1\leqslant p\leqslant\infty)$ if the sequence $(Q_a^k\phi_0)_{k\geqslant 1}$ converges in $W_p^n(\mathbb{R}^s)$-norm. In this case, it has been proved in [3] that ϕ_0 is necessarily contained in the space

$$W_n := \{f \in W_p^n(\mathbb{R}^s) \text{ compactly supp.} : D^\mu \hat{f}(2\pi\alpha) = 0 \ \forall \alpha \in \mathbb{Z}^s\setminus\{0\},\ |\mu|\leqslant n\}. \tag{1.5}$$

The cascade operator Q_a is closely connected with the *subdivision operator* $S_a : \ell_0(\mathbb{Z}^s) \to \ell_0(\mathbb{Z}^s)$ *associated with the mask a*,

$$S_a v(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta) v(\beta), \qquad \alpha \in \mathbb{Z}^s.$$

Denoting $a_k := S_a^k\delta$, where δ is the impulse sequence given by $\delta(\alpha) = 0$ for $\alpha \in \mathbb{Z}^s\setminus\{0\}$ and $\delta(0) = 1$, we have $a_1 = a$ and

$$a_k(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a_{k-1}(\beta) a(\alpha - M\beta), \qquad \alpha \in \mathbb{Z}^s, \quad k\geqslant 2. \tag{1.6}$$

It can easily be verified by induction (see [11]) that for $f \in L_p(\mathbb{R}^s)$

$$Q_a^k f = \sum_{\alpha\in\mathbb{Z}^s} a_k(\alpha) f(M^k\cdot -\alpha), \qquad k = 1, 2, \ldots\ . \tag{1.7}$$

The cascade algorithm plays an important role in computer graphics and wavelet analysis. The convergence of the cascade algorithm has been studied by many authors. Cavaretta *et al.* [1] already found necessary and sufficient conditions ensuring that the subdivision scheme related to a finitely supported refinement mask with dilation matrix $M = 2I$ uniformly converges to a continuous limit function. In the L_2-norm, the convergence of the cascade algorithm has been shown by Strang [28] in the univariate case, by Lawton *et al.* [23] in the multivariate case and by Shen [27] in the general multivariate vector case. Jia [15] considered the convergence of subdivision schemes in the univariate setting for general L_p-spaces; the multivariate L_p-case is completely settled in Han and Jia [11]. For the univariate vector case, we refer to Jia *et al.* [21] and to Micchelli and Sauer [24, 25]. Convergence in $W_2^n(\mathbb{R}^s)$ has firstly been discussed by Jia, *et al.* [19].

For scalar subdivision schemes in Sobolev spaces, we also refer to Goodman and Lee [6] and to Micchelli and Sauer [26]. The cascade algorithm in Besov spaces has been considered by Sun [29]. Chen *et al.* [3] and Zhou [31] have studied this problem in $W_p^n(\mathbb{R}^s)$ for $1 \leqslant p \leqslant \infty$.

In practice, one often has to handle perturbed refinement masks. In fact, coefficients are generally irrational or rational numbers which need to be truncated in floating point arithmetics. Heil and Collela [12] were the first, who studied how such truncation affects the refinable function in the univariate L_∞-case (see also [13]). Further discussions on the effect of perturbed scaling coefficients in the univariate case can be found in [4, 30]. Villemoes even showed that, under certain conditions, membership of a refinable function in a Besov class is stable under perturbations.

More recently, Han [7, 8] provided a sharp error estimate for multivariate refinable functions in any L_p-norm. His idea has been adopted to perturbed matrix masks in the univariate L_p-case by Han and Hogan [10].

In particular, Han could show the following result in [7, 8]: If the cascade algorithm related to a mask a converges for ϕ_0 in L_p-norm, and if b is an only slightly perturbed mask, i.e., $\|a-b\|_1 < \eta$ for a sufficiently small $\eta > 0$, and b satisfies the sum rules of order 1 (see Section 2 for the notion of sum rules of order n), then the cascade algorithm associated with b also converges for ϕ_0 in L_p-norm and we have

$$\|Q_a^k\phi_0 - Q_b^k\phi_0\|_p \leqslant C\|a-b\|_1, \qquad k \geqslant 1. \tag{1.8}$$

Here the constant C depends on the refinement mask a under consideration as well as on p, $1 \leqslant p \leqslant \infty$. However, it is independent of the perturbed masks b and k.

In this paper, we want to generalize the above result to cascade algorithms converging in Sobolev spaces.

Compared with the L_p-case, we have to overcome some difficulties due to the handling with function derivatives requiring another approach. In fact, the proof of the main result is based on two new key ingredients.

The first basic idea to obtain the wanted estimate is the observation that for some suitable initial function ϕ_0 the following inequality holds: There exists a positive constant c with

$$\sum_{|\mu|=n} \|D^\mu Q_a^k\phi_0\|_p \leqslant c m^{(n/s-1/p)k} \sum_{|\mu|=n} \|\Delta^\mu a_k\|_p$$

for all $k = 1, 2, \ldots$ (see Theorem 3.2). Here Δ^μ denotes the μth difference operator (see Section 2) and a_k is the iterated subdivision operator applied to δ in (1.6).

The second key ingredient for the wanted estimate is the inequality

$$\|\Delta^\mu a_k - \Delta^\mu b_k\|_p \leqslant c\|a-b\|_1 m^{(-n/s+1/p)k} \qquad \forall |\mu| = n,\ k = 1,2,\ldots$$

(see Lemma 4.3). The proof of this inequality requires exact analysis of the connection between convergence and boundedness of $(Q_a^k\phi_0)_{k\geqslant 0}$ (resp. $(Q_b^k\phi_0)_{k\geqslant 0}$) and the behavior of $\|\Delta^\mu a_k\|_p$ (resp. $\|\Delta^\mu b_k\|_p$) with $|\mu| = n$, even slightly extending the known results on convergence of cascade algorithms in Sobolev spaces (see [3]).

In Section 2, we recall some important definitions and results from [3,11]. In particular, two equivalent characterizations of the convergence of cascade algorithms in $W_p^n(\mathbb{R}^s)$ are given in terms of the joint spectral radius and of the subdivision operator. In Section 3, we construct a special initial function satisfying the above useful inequality. Further, an implicit relation between the boundedness and convergence of a cascade algorithm in different Sobolev spaces is established. Section 4 is devoted to the generalization of (1.8) to Sobolev spaces.

2. JOINT SPECTRAL RADII

In the study of convergence of the cascade algorithm, the joint spectral radius of linear operators is a useful tool. The uniform joint spectral radius was employed in [5] to investigate the regularity of refinable functions. For $1 \leqslant p < \infty$, the p-joint spectral radius was introduced and applied to the study of L_p-convergence of cascade algorithms by Jia [15]. We cite from [15] the definition of p-norm joint spectral radius for the convenience of the reader.

Let V be a finite-dimensional space with norm $\|\cdot\|$. For a linear operator A on V define

$$\|A\| := \max\{\|Av\| : \|v\| = 1\}.$$

Let $\mathscr{A}$ be a finite collection of linear operators on a finite-dimensional vector space V. For a positive integer k, we denote by $\mathscr{A}^k$ the Cartesian power of $\mathscr{A}$:

$$\mathscr{A}^k = \{(A_1,\ldots,A_k) : A_1,\ldots,A_k \in \mathscr{A}\}.$$

Now let

$$\|\mathscr{A}^k\|_p := \begin{cases} \left(\sum_{(A_1,\ldots,A_k)\in\mathscr{A}^k} \|A_1\cdots A_k\|^p\right)^{1/p}, & 1\leqslant p<\infty,\\ \max\{\|A_1\cdots A_k\| : (A_1,\ldots,A_k)\in\mathscr{A}^k\}, & p=\infty. \end{cases}$$

The p-norm joint spectral radius of $\mathscr{A}$ is defined to be

$$\rho_p(\mathscr{A}) := \lim_{k\to\infty} \|\mathscr{A}^k\|_p^{1/k}. \tag{2.1}$$

This limit indeed exists and does not depend on the choice of norm on V. Moreover, we have

$$\lim_{k\to\infty} \|\mathscr{A}^k\|_p^{1/k} = \inf_{k\geqslant 1} \|\mathscr{A}^k\|_p^{1/k}. \tag{2.2}$$

Further, let for $v \in V$

$$\|\mathscr{A}^k v\|_p := \begin{cases} \left(\sum_{(A_1,\dots,A_k)\in\mathscr{A}^k} \|A_1\cdots A_k v\|^p\right)^{1/p}, & 1\leqslant p<\infty, \\ \max\{\|A_1\cdots A_k v\| : (A_1,\dots,A_k)\in\mathscr{A}^k\}, & p=\infty. \end{cases}$$

Let us come back to our problem. For a finite refinement mask a, we consider m operators $A_\varepsilon, \varepsilon\in E$, on $\ell_0(\mathbb{Z}^s)$ defined by the biinfinite matrices

$$A_\varepsilon(\alpha,\beta) = a(\varepsilon + M\alpha - \beta), \qquad \alpha,\beta\in\mathbb{Z}^s. \tag{2.3}$$

Hence,

$$A_\varepsilon v(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(\varepsilon + M\alpha - \beta)v(\beta), \qquad v\in\ell_0(\mathbb{Z}^s). \tag{2.4}$$

Now, let $\mathscr{A}$ be the finite collection of A_ε, $\varepsilon\in E$. There is a simple relation between a_k in (1.6) and the matrices $A_\varepsilon, \varepsilon\in E$ in (2.3). Let $\alpha\in\mathbb{Z}^s$ and k be a positive integer. Then there are (uniquely defined) $\varepsilon_1,\dots,\varepsilon_k\in E$ and $\gamma\in\mathbb{Z}^s$ such that $\alpha = \varepsilon_1 + M\varepsilon_2 + \cdots + M^{k-1}\varepsilon_k + M^k\gamma$ and we have (see [11, Lemma 2.1])

$$a_k(\alpha-\beta) = A_{\varepsilon_k}\cdots A_{\varepsilon_1}(\gamma,\beta) \qquad \forall\beta\in\mathbb{Z}^s. \tag{2.5}$$

For two sequences $u\in\ell_p(\mathbb{Z}^s)$ and $v\in\ell_0(\mathbb{Z}^s)$, the discrete convolution $u*v\in\ell_p(\mathbb{Z}^s)$ is defined by

$$(u*v)(\alpha) = \sum_{\beta\in\mathbb{Z}^s} u(\alpha-\beta)v(\beta), \qquad \alpha\in\mathbb{Z}^s.$$

It follows from equality (2.5) that, for any $v\in\ell_0(\mathbb{Z}^s)$,

$$(a_k*v)(\alpha) = A_{\varepsilon_k}\cdots A_{\varepsilon_1}v(\gamma), \tag{2.6}$$

with $\alpha = \varepsilon_1 + M\varepsilon_2 + \cdots + M^{k-1}\varepsilon_k + M^k\gamma$ and consequently for $1 \leqslant p < \infty$,

$$\|a_k * v\|_p^p = \sum_{\varepsilon_1,\dots,\varepsilon_k \in E} \|A_{\varepsilon_k} \cdots A_{\varepsilon_1} v\|_p^p = \|\mathscr{A}^k v\|_p^p.$$

Let e_j be the jth coordinate unit vector of $\mathbb{R}^s$, $j = 1, 2, \dots, s$. Recall that for any $j = 1, 2, \dots, s$ and a function f defined on $\mathbb{R}^s$, the difference operator Δ_j is given by

$$\Delta_j f \coloneqq f(\cdot) - f(\cdot - e_j).$$

Analogously, let the difference operator Δ_j be defined for sequences $\lambda \in \ell(\mathbb{Z}^s)$, by $\Delta_j \lambda = \lambda(\cdot) - \lambda(\cdot - e_j)$. Further, for any $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{Z}_+^s$, denote $\Delta_1^{\mu_1} \cdots \Delta_s^{\mu_s}$ by Δ^μ.

In order to give a characterization for convergence of the cascade algorithm in $W_p^n(\mathbb{R}^s)$-norm, we introduce the subspace

$$V_n \coloneqq \left\{ v = (v(\alpha))_{\alpha \in \mathbb{Z}^s} \in \ell_0(\mathbb{Z}^s) : \sum_{\alpha \in \mathbb{Z}^s} \alpha^\mu v(\alpha) = 0, \ |\mu| \leqslant n \right\}. \tag{2.7}$$

Observe that $V_n = \operatorname{span}\{\Delta^\mu \delta(\cdot - \beta) : \beta \in \mathbb{Z}^s, \ |\mu| = n+1\}$, where δ is the impulse sequence. Furthermore, one can construct a finite set $K \subseteq \mathbb{Z}^s$ such that $\ell(K)$ is a finite subspace of $\ell_0(\mathbb{Z}^s)$ consisting of all sequences with support on K with the following properties:

1. $\ell(K)$ is an invariant subspace under A_ε for any $\varepsilon \in E$;
2. $\ell(K)$ contains $\Delta^\mu \delta, |\mu| = n+1$.

To this end, let $\operatorname{supp} a \coloneqq \{\alpha : a(\alpha) \neq 0\}$ and Ω be a finite set of $\mathbb{Z}^s$ such that $\operatorname{supp} a \cup \{0\} \subseteq \Omega$. Put $H \coloneqq \Omega - E + M\mathbb{Z}_{n+1}^s$, where $\mathbb{Z}_{n+1}^s \coloneqq \{(\mu_1, \dots, \mu_s) \in \mathbb{Z}^s, \ 0 \leqslant \mu_i \leqslant n+1, \ 1 \leqslant i \leqslant s\}$. (Here, the set $A + B$ (or $A - B$) consists of all points $x + y$ (or $x - y$) with $x \in A$ and $y \in B$.) Now, let

$$K \coloneqq \mathbb{Z}^s \cap \sum_{k=1}^{\infty} M^{-k} H. \tag{2.8}$$

In particular, we have $M^{-1}(K + \Omega - E) \cap \mathbb{Z}^s \subseteq K$. It is not difficult to see that $\ell(K)$ is invariant under $A_\varepsilon, \varepsilon \in E$, i.e. for $v \in \ell(K)$ we have $A_\varepsilon v \in \ell(K)$ (see [11, Lemma 2.3]).

Then, in [3] the following has been shown:

Result 2.1 (Chen *et al.* [3]). Let $a \in \ell_0(\mathbb{Z}^s)$ satisfy (1.3) and let W_n be given in (1.5). The cascade algorithm associated with a converges for all functions ϕ in W_n in $W_p^n(\mathbb{R}^s)$-norm $(1 \leqslant p \leqslant \infty)$ if and only if the following conditions are satisfied:

(1) V_n is invariant under $A_\varepsilon\ \forall \varepsilon \in E$, i.e. for $v \in V_n$ it follows that $A_\varepsilon v \in V_n$;
(2) $\rho_p(\{A_\varepsilon|_{V_n \cap \ell(K)} : \varepsilon \in E\}) < m^{-n/s+1/p}$, where K is given in (2.8).

Remark. (1) Condition (1) in Result 2.1 is a necessary condition on the mask a, it needs to be satisfied if the limit function of cascade algorithm is wanted to be in $W_p^n(\mathbb{R}^s)$. Moreover, (1) is equivalent with the *sum rules of order* $n+1$, saying that for any $p \in \Pi_n$

$$\sum_{\alpha \in \mathbb{Z}^s} p(M\alpha + \varepsilon) a(M\alpha + \varepsilon) = \sum_{\alpha \in \mathbb{Z}^s} p(M\alpha) a(M\alpha) \qquad \forall \varepsilon \in E. \tag{2.9}$$

This equivalence has already been shown in [18, Theorem 5.2] (see also [14, Theorem 3.4.12]). We want to remark that condition (1), or equivalently, the sum rules of order $n+1$ are also necessary for reproduction of polynomials up to total degree n in the shift-invariant space $S(\phi)$ generated by the integer translates of the M-refinable function ϕ (see [2, 17]).

(2) Condition (2) in Result 2.1 can be seen as a generalization of the result in [11], where the convergence of cascade algorithms in L_p-spaces is shown.

Since K is a finite set, the p-norm joint spectral radius needs to be determined only in the finite-dimensional space $V_n \cap \ell(K)$.

The following lemma justifies the definition of the set K in (2.8). Here, we consider the action of operators $A_\varepsilon, \varepsilon \in E$, on the sequences with supports contained in any fixed finite set $K_1 \subseteq \mathbb{Z}^s$.

Lemma 2.2. *Let K be defined by* (2.8). *Then for any finite set $K_1 \subseteq \mathbb{Z}^s$, there is a positive integer j such that*

$$A_{\varepsilon_j} \cdots A_{\varepsilon_1} v \in \ell(K) \qquad \forall v \in \ell(K_1) \quad \textit{and} \quad \varepsilon_1, \ldots, \varepsilon_j \in E. \tag{2.10}$$

Consequently, for any integer $k > j$

$$A_{\varepsilon_k} \cdots A_{\varepsilon_1} v \in \ell(K) \qquad \forall v \in \ell(K_1) \quad \textit{and} \quad \varepsilon_1, \ldots, \varepsilon_k \in E. \tag{2.11}$$

Proof. For any $v \in \ell(K_1)$, we have $\operatorname{supp} A_\varepsilon v \subset M^{-1}(K_1 + \Omega - E) \cap \mathbb{Z}^s$, $\varepsilon \in E$. Iterative application yields for any integer $j > 0$

$$\operatorname{supp} A_{\varepsilon_j} \cdots A_{\varepsilon_1} v \subseteq (M^{-j}(K_1 + \Omega - E) \cap \mathbb{Z}^s)$$
$$+ (M^{-j+1}(\Omega - E) \cap \mathbb{Z}^s) + \cdots + (M^{-1}(\Omega - E) \cap \mathbb{Z}^s),$$

where Ω contains the support of a.

Since M is isotropic, there is a constant c being independent of j such that

$$\|M^{-j}\omega\| \leqslant cm^{-j/s}\|\omega\| \qquad \forall \omega \in \mathbb{R}^s \quad \text{and} \quad j = 1, 2, \ldots \tag{2.12}$$

with $m = |\det M| > 1$ (see e.g. [17, Lemma 6.1]). Therefore, we can find an integer j such that, for all $\alpha \in K_1 + \Omega - E$, $M^{-j}\alpha \in (-1, 1)^s$, i.e. $M^{-j}(K_1 + \Omega - E) \cap \mathbb{Z}^s \in \{\emptyset, \{0\}\}$ and (2.10) holds. Since $\ell(K)$ is an invariant subspace under A_ε for any $\varepsilon \in E$, (2.11) follows for any $k > j$. ∎

There is a second way to characterize the convergence of the cascade algorithm using the subdivision operator S_a.

THEOREM 2.3. *Let* $a \in \ell_0(\mathbb{Z}^s)$ *satisfy* (1.3). *Then the cascade algorithm associated with a converges for all functions in* W_n *in* $W_p^n(\mathbb{R}^s)$*-norm* $(1 \leqslant p \leqslant \infty)$ *if and only if*

$$\lim_{k \to \infty} m^{k(n/s - 1/p)} \|\Delta^\mu a_k\|_p = 0 \qquad \forall |\mu| = n + 1, \tag{2.13}$$

where $a_k = S_a^k \delta$.

The proof of this theorem will be given in the next section.

3. DIFFERENTIAL AND DIFFERENCE OPERATOR

We now turn our attention to the norms $\|Q_a^k \phi_0\|_{W_p^n(\mathbb{R}^s)}$. The goal is to estimate them in terms of sequence norms deduced from a_k. In particular, we shall show in this section that boundedness of $(Q_a^k \phi_{0,n})_{k \geqslant 1}$ (where $\phi_{0,n}$ is a suitably chosen initial function in W_n) implies convergence of the cascade algorithm on W_{n-1} in $W_p^{n-1}(\mathbb{R}^s)$-norm.

Let f be a differentiable function on $\mathbb{R}^s$ and let $\mathscr{D} := (D_1, \ldots, D_s)^T$ with $D_j = \frac{\partial}{\partial x_j}$. Then, the chain rule for differentiation gives

$$\mathscr{D}(f(M^k \cdot))(x) = (M^T)^k \mathscr{D} f(M^k x), \qquad x \in \mathbb{R}^s,$$

where M^T is the transpose of M. Since M is isotropic, there exists an invertible matrix Λ such that $\Lambda M^T \Lambda^{-1} = \operatorname{diag}(\sigma_1, \ldots, \sigma_s)$. Hence, we have

$$\Lambda \mathscr{D}(f(M^k \cdot))(x) = \operatorname{diag}(\sigma_1^k, \ldots, \sigma_s^k) \Lambda \mathscr{D} f(M^k x).$$

Let $q_j(\mathscr{D}) := \Lambda_j \mathscr{D}$, where Λ_j denotes the jth row of Λ $(j = 1, \ldots, s)$, and for any $\mu = (\mu_1, \ldots, \mu_s)^T \in \mathbb{Z}_+^s$, let $q_\mu(\mathscr{D}) := q_1(\mathscr{D})^{\mu_1} \cdot \ldots \cdot q_s(\mathscr{D})^{\mu_s}$.

Considering the last equation componentwisely, we have for any $f \in W_p^n(\mathbb{R}^s)$

$$q_j(\mathscr{D})(f(M^k\cdot))(x) = \sigma_j^k(q_j(\mathscr{D})f)(M^k x), \qquad j = 1, \ldots, s,$$

and hence

$$q_\mu(\mathscr{D})(f(M^k\cdot))(x) = (\sigma_1^{\mu_1 k} \cdot \ldots \cdot \sigma_s^{\mu_s k})(q_\mu(\mathscr{D})f)(M^k x), \qquad x \in \mathbb{R}^s \tag{3.1}$$

(see also [19, 22, 31]).

It is easily seen that the operators $q_\mu(\mathscr{D})$ may be expressed as

$$q_\mu(\mathscr{D}) = \sum_{|v|=|\mu|} c_{\mu,v} D^v,$$

where $c_{\mu,v}$ are determined by Λ and $D^v = D_1^{v_1} \ldots D_s^{v_s}$. Since Λ is invertible, there exists a positive number κ satisfying, for any $f \in W_p^n(\mathbb{R}^s)$,

$$\kappa^{-1} \sum_{|\mu|=n} |(D^\mu f)(x)| \leqslant \sum_{|\mu|=n} |q_\mu(D)f(x)| \leqslant \kappa \sum_{|\mu|=n} |(D^\mu f)(x)|, \qquad x \in \mathbb{R}^s.$$

Applying this equivalence and (3.1), we find for any $f \in W_p^n(\mathbb{R}^s)$

$$\kappa^{-1} m^{nk/s} \sum_{|\mu|=n} |(D^\mu f)(M^k x)| \leqslant \sum_{|\mu|=n} |D^\mu(f(M^k\cdot))(x)|$$

$$\leqslant \kappa m^{nk/s} \sum_{|\mu|=n} |(D^\mu f)(M^k x)|, \qquad x \in \mathbb{R}^s \quad \text{and} \quad k = 1, 2, \ldots,$$

where we have used that $|\sigma_1| = \cdots = |\sigma_s| = m^{1/s}$. The second inequality has been also proved in [17]. Hence, we obtain

LEMMA 3.1. *There is a positive number c such that for any nontrivial* $f \in W_p^n(\mathbb{R}^s)$

$$c^{-1} m^{(n/s-1/p)k} \leqslant \frac{\sum_{|\mu|=n} \|D^\mu(f(M^k\cdot))\|_p}{\sum_{|\mu|=n} \|D^\mu f\|_p} \leqslant c m^{(n/s-1/p)k}, \qquad k = 1, 2, \ldots .$$

In these inequalities, the factor $m^{-k/p}$ is due to the change of variables $M^k x \to x$ in the norms.

For our considerations, we want to use a special initial function ϕ_0 which is a tensor product of univariate B-splines. For $k \in \mathbb{Z}_+$, let N_k be the univariate forward B-spline of degree k with the knots $0, 1, \ldots, k+1$, recursively given by

$$N_k = N_{k-1} * N_0 = \int_0^1 N_{k-1}(\cdot - t)\,dt, \qquad t \in \mathbb{R},$$

where $N_0 := \chi_{[0,1)}$ is the characteristic function of [0,1). Furthermore, for $v = (v_1, \ldots, v_s) \in \mathbb{Z}_+^s$, let $N_v(x) := N_{v_1}(x_1) \cdots N_{v_s}(x_s)$, where $x = (x_1, \ldots, x_s)^T \in \mathbb{R}^s$.

Observe that for any pair of μ and $v \in \mathbb{Z}_+^s$ with $\mu \leqslant v$

$$D^\mu N_v = \Delta^\mu N_{v-\mu}. \tag{3.2}$$

A second important property of N_v in this context is the *stability* of its shifts. This means that, for any $v \in \mathbb{Z}_+^s$, there is a positive number κ, which is independent of λ, satisfying

$$\kappa^{-1} \|\lambda\|_p \leqslant \sum_{\alpha \in \mathbb{Z}^s} \|\lambda(\alpha) N_v(\cdot - \alpha)\|_p \leqslant \kappa \|\lambda\|_p \qquad \forall \lambda \in \ell_p(\mathbb{Z}^s). \tag{3.3}$$

The functions N_v are appropriate candidates for the initial function in the cascade algorithm. In fact,

$$\phi_{0,n} = N_{(n+1,\ldots,n+1)} \tag{3.4}$$

is in W_n for any $1 \leqslant p \leqslant \infty$ (with W_n in (1.5)).

THEOREM 3.2. *Let $\lambda \in \ell_0(\mathbb{Z}^s)$ and let g be associated with λ by*

$$g = \sum_{\alpha \in \mathbb{Z}^s} \lambda(\alpha) \phi_{0,n}(M^k \cdot - \alpha).$$

Then, there exists a constant $\kappa > 0$ which is independent of $\lambda \in \ell_0(\mathbb{Z}^s)$ and $k \in \mathbb{Z}_+^s$, such that

$$\kappa^{-1} m^{(n/s-1/p)k} \leqslant \frac{\sum_{|\mu|=n} \|D^\mu g\|_p}{\sum_{|\mu|=n} \|\Delta^\mu \lambda\|_p} \leqslant \kappa m^{(n/s-1/p)k}, \qquad k = 1, 2, \ldots . \tag{3.5}$$

In particular, if the sequence $(Q_a^k \phi_{0,n})_{k \geqslant 1}$ is bounded in $W_p^n(\mathbb{R}^s)$, then there is a constant c being independent of k such that

$$m^{k(n/s-1/p)} \|\Delta^\mu a_k\|_p \leqslant c \qquad \forall |\mu| = n, \quad k = 1, 2, \ldots . \tag{3.6}$$

Further, if the sequence $(Q_a^k \phi_{0,n})_{k \geqslant 1}$ converges in $W_p^n(\mathbb{R}^s)$, then (2.13) *holds.*

Proof. For $\lambda = a_k = S_a^k \delta$, the function g associated with λ equals to $Q_a^k \phi_{0,n}$ by (1.7). If the sequence $(Q_a^k \phi_{0,n})_{k \geqslant 1}$ is bounded in $W_p^n(\mathbb{R}^s)$-norm, then

(3.6) follows from the first inequality in (3.5). If the sequence $(Q_a^k\phi_{0,n})_{k\geqslant 1}$ converges in $W_p^n(\mathbb{R}^s)$, then there exists a compactly supported limit function $\phi_a \in W_p^n(\mathbb{R}^s)$ such that $\|Q_a^k\phi_{0,n} - \phi_a\|_{W_p^n(\mathbb{R}^s)} \to 0$ for $k \to \infty$. Further, from

$$\|Q_a^k\phi_{0,n} - Q_a^k\phi_{0,n}(\cdot - M^{-k}e_j)\|_{W_p^n(\mathbb{R}^s)}$$
$$\leqslant \|\phi_a - \phi_a(\cdot - M^{-k}e_j)\|_{W_p^n(\mathbb{R}^s)} + 2\|\phi_a - Q_a^k\phi_{0,n}\|_{W_p^n(\mathbb{R}^s)}$$

for all unit vectors e_j, $j = 1, \ldots, s$ we obtain that

$$\sum_{|\mu|=n} \|\Delta_j D^\mu Q_a^k\phi_{0,n}\|_p \to 0 \qquad \text{for} \quad k \to \infty,\ j = 1, \ldots, s.$$

Now again for $\lambda = a_k$, we have $g = Q_a^k\phi_{0,n}$ and (2.13) follows from the first inequality of (3.5) as before.

Let us now prove (3.5). Putting $f = g(M^{-k}\cdot)$, we obtain by (3.2)

$$D^\mu f = \sum_{\alpha\in\mathbb{Z}^s} \lambda(\alpha)\Delta^\mu N_v(\cdot - \alpha) = \sum_{\alpha\in\mathbb{Z}^s} \Delta^\mu\lambda(\alpha)N_v(\cdot - \alpha),$$

where $v = (n + 1 - \mu_1, \ldots, n + 1 - \mu_s)$. Consequently,

$$D^\mu g = (D^\mu f)(M^k\cdot) = \sum_{\alpha\in\mathbb{Z}^s} \Delta^\mu\lambda(\alpha)N_v(M^k \cdot - \alpha).$$

Therefore, the inequalities in (3.5) are true by Lemma 3.1 and the stability property (3.3) of N_v. ∎

Remark. The necessity of (2.13) for the convergence of the cascade algorithm in $W_p^n(\mathbb{R}^s)$ has also been shown in [3, Lemma 4.3].

Now we are able to show the following relation.

LEMMA 3.3. *Assume that* (2.13) *is true for a given refinement mask a. Then V_n in* (2.7) *is an invariant subspace under A_ε for all $\varepsilon \in E$.*

Proof. For $\varepsilon \in E$ and $\mu \in \mathbb{Z}_+^s$, we define a polynomial $p_{\varepsilon,\mu} \in \Pi_{|\mu|}$ by

$$p_{\varepsilon,\mu}(x) = \sum_{\beta\in\mathbb{Z}^s} a(M\beta + \varepsilon)(M^{-1}(x - \varepsilon) - \beta)^\mu.$$

The space V_n is invariant under A_ε for all $\varepsilon \in E$ if and only if the mask a satisfies the sum rules of order $n + 1$ in (2.9). Hence, we have to show

$$p_{\varepsilon_1,\mu} = p_{\varepsilon_2,\mu} \qquad \forall\varepsilon_1, \varepsilon_2 \in E \quad \text{and} \quad \forall\mu \text{ with } |\mu| \leqslant n. \tag{3.7}$$

For $|\mu| = 0$, (3.7) has been proved in [11, Theorem 3.1]. We shall prove (3.7) by induction on n_0 with $0 \leqslant n_0 \leqslant n$. Assume that (3.7) holds for $n_0 < n$. If it is not true for $n_0 + 1$, then there are $\varepsilon_1, \varepsilon_2 \in E$ and $\mu \in \mathbb{Z}_+^s$ with $|\mu| = n_0 + 1$ such that $p_{\varepsilon_1,\mu} \neq p_{\varepsilon_2,\mu}$. We shall show that this contradicts (2.13).

For any $\mu \in \mathbb{Z}_+^s$ and $k = 1, 2, \ldots$, let $h_{k,\mu} \in \ell_0(\mathbb{Z}^s)$ be defined by

$$h_{k,\mu}(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a_k(\alpha - M\beta)\beta^\mu, \qquad \alpha \in \mathbb{Z}^s,$$

where a_k are given in (1.6). Observe that for any $\varepsilon \in E$

$$h_{1,\mu}(M\alpha + \varepsilon) = p_{\varepsilon,\mu}(M\alpha + \varepsilon) \qquad \forall \alpha \in \mathbb{Z}^s.$$

Thus, the induction assumption (3.7) for n_0 implies that

$$h_{1,\mu}(\alpha) = p_{\varepsilon,\mu}(\alpha) \qquad \forall \mu, |\mu| \leqslant n_0, \quad \forall \varepsilon \in E \quad \text{and} \quad \alpha \in \mathbb{Z}^s.$$

Consequently, since $p_{\varepsilon,\mu} \in \Pi_{|\mu|}$ we have $\Delta^\gamma h_{1,\mu} = \Delta^\gamma p_{\varepsilon,\mu} = 0$ for $|\gamma| = |\mu| + 1$ and $|\mu| \leqslant n_0$, i.e., $h_{1,\mu}$ ($|\mu| \leqslant n_0$) is a polynomial sequence of degree $|\mu|$.

Now, let $|\mu| = n_0 + 1$. Since by assumption $p_{\varepsilon_1,\mu} \neq p_{\varepsilon_1,\mu}$ for some $\varepsilon_1, \varepsilon_2 \in E$ and some $|\mu| = n_0 + 1$, we have $h_{1,\mu}(M\alpha + \varepsilon_1) = p_{\varepsilon_1,\mu}(M\alpha + \varepsilon_1)$ $\forall \alpha \in \mathbb{Z}^s$ but $h_{1,\mu}(M\alpha + \varepsilon_2) \neq p_{\varepsilon_1,\mu}(M\alpha + \varepsilon_2)$ for some $\alpha \in \mathbb{Z}^s$. Hence, $h_{1,\mu}(\alpha)$ cannot be a polynomial sequence of degree $n_0 + 1$, i.e., there exist $\gamma_0 \in \mathbb{Z}^s$, $|\gamma_0| = n_0 + 2$ and $\alpha \in \mathbb{Z}^s$ such that

$$\Delta^{\gamma_0} h_{1,\mu}(\alpha) \neq 0. \tag{3.8}$$

On the other hand, relation (1.6) tells us that for $\alpha \in \mathbb{Z}^s$

$$\begin{aligned} h_{k,\mu}(\alpha) &= \sum_{\beta \in \mathbb{Z}^s} \sum_{\delta \in \mathbb{Z}^s} a(\alpha - M\beta - M\delta) a_{k-1}(\delta)(\beta + \delta - \delta)^\mu \\ &= \sum_{\nu \leqslant \mu} \binom{\mu}{\nu} (-1)^{|\mu-\nu|} h_{1,\nu}(\alpha) \sum_{\delta \in \mathbb{Z}^s} a_{k-1}(\delta)\delta^{\mu-\nu} \\ &= \sum_{\nu < \mu} \binom{\mu}{\nu} (-1)^{|\mu-\nu|} h_{1,\nu}(\alpha) \sum_{\delta \in \mathbb{Z}^s} a_{k-1}(\delta)\delta^{\mu-\nu} + h_{1,\mu}(\alpha) \sum_{\delta \in \mathbb{Z}^s} a_{k-1}(\delta). \end{aligned}$$

Since $\sum_{\delta \in \mathbb{Z}^s} a(\delta) = m$ (see (1.3)), a simple induction argument gives $\sum_{\delta \in \mathbb{Z}^s} a_{k-1}(\delta) = m^{k-1}$. Thus,

$$\Delta^{\gamma_0} h_{k,\mu}(\alpha) = m^{k-1} \Delta^{\gamma_0} h_{1,\mu}(\alpha) \qquad \forall \alpha \in \mathbb{Z}^s. \tag{3.9}$$

It is easily seen by induction that $\operatorname{supp} \Delta^\gamma a_k \subseteq \{\alpha \in \mathbb{Z}^s : \|\alpha\|_\infty \leqslant \kappa m^{k/s}\}$ for some constant κ independent of $k = 1, 2, \ldots$. For any fixed $\alpha \in \mathbb{Z}^s$, let $\Gamma_k = \Gamma_k(\alpha) := \mathbb{Z}^s \cap M^{-1}(\alpha - \operatorname{supp} \Delta^\gamma a_k)$, i.e., Γ_k denotes the support of $\Delta^\gamma a_k(\alpha - M\cdot)$. Then, the cardinality of Γ_k satisfies

$$\#\Gamma_k \leqslant \kappa' m^k, \qquad k = 1, 2, \ldots,$$

where κ' is a constant. For $|\mu| = n_0 + 1$ and $|\gamma| = n_0 + 2$, it follows from Hölder's inequality that

$$\begin{aligned}
|\Delta^\gamma h_{k,\mu}(\alpha)| &= \left| \sum_{\beta \in \Gamma_k} \beta^\mu \Delta^\gamma a_k(\alpha - M\beta) \right| \\
&\leqslant \left(\sum_{\beta \in \Gamma_k} \|\beta\|_\infty^{(n_0+1)q} \right)^{1/q} \left(\sum_{\beta \in \Gamma_k} |\Delta^\gamma a_k(\alpha - M\beta)|^p \right)^{1/p} \\
&\leqslant c_1 m^{k(n_0+1)/s} m^{k/q} \|\Delta^\gamma a_k\|_p
\end{aligned}$$

for some constant c_1 dependent of α but not of $k = 1, 2, \ldots$, where q satisfies $1/p + 1/q = 1$. This together with (2.13) and (3.9) gives us that $\left|\Delta^{\gamma_0} h_{1,\mu}(\alpha)\right|$ tends to zero for $k \to \infty$, in contradiction with (3.8). This completes the induction process, thereby proving the assertion. ∎

Proof of Theorem 2.3. The necessity of (2.13) for convergence of the cascade algorithm has already been shown in Theorem 3.2. In order to show sufficiency, we need to prove that conditions (1) and (2) of Result 2.1 follow from (2.13). By Lemma 3.3, the sum rules of order $n+1$ are satisfied. Further, by (2.6) and the definition of V_n in (2.7) we have for $|\mu| = n + 1$

$$\lim_{k \to \infty} \|\Delta^\mu a_k\|_p^{1/k} = \lim_{k \to \infty} \|\mathscr{A}^k \Delta^\mu \delta\|_p^{1/k} = \rho_p(\{A_\varepsilon|_{V_n \cap \ell(K)},\ \varepsilon \in E\})$$

(see [11, Theorem 2.5]). Hence, the assertion follows. ∎

Finally, we obtain

COROLLARY 3.4. *Assume that the sequence* $(Q_a^k \phi_{0,n})_{k \geqslant 1}$ *is bounded in* $W_p^n(\mathbb{R}^s)$*-norm. Then, the cascade algorithm corresponding to mask a converges for every* $\phi \in W_{n-1}$ *in* $W_p^{n-1}(\mathbb{R}^s)$*-norm.*

Proof. Comparing (3.6) with (2.13) (for $n-1$ instead of n), the assertion directly follows from Theorem 2.3. ∎

4. PERTURBATIONS OF REFINEMENT MASKS

In this section, we shall show the convergence of the cascade algorithm corresponding to a slightly perturbed refinement mask. Moreover, the perturbation of the refinable limit function affected by the perturbation of refinement mask is studied.

THEOREM 4.1. *Let Ω be a finite set of $\mathbb{Z}^s$. Assume that the cascade algorithm corresponding to $a \in \ell(\Omega)$ converges for every $\phi \in W_n$ in $W_p^n(\mathbb{R}^s)$-norm. Then there is a positive number η such that, for any $b \in \ell(\Omega)$ satisfying* (1.3), *sum rules of order $n+1$ and $\|a-b\|_1 < \eta$, the cascade algorithm corresponding to b also converges for every $\phi \in W_n$ in $W_p^n(\mathbb{R}^s)$-norm.*

Proof. Recall that K is defined in (2.8). By assumption on a, it follows from Result 2.1, that

$$\lim_{k\to\infty} \|\mathscr{A}^k|_{V_n \cap \ell(K)}\|_p^{1/k} = \inf_{k \geqslant 1} \|\mathscr{A}^k|_{V_n \cap \ell(K)}\|_p^{1/k} < m^{-n/s+1/p}.$$

Hence, there exists an integer $k \geqslant 1$ and some positive t such that

$$\max_{\substack{v \in V_n \cap \ell(K) \\ \|v\|=1}} \sum_{\varepsilon_1,\ldots,\varepsilon_k \in E} \|A_{\varepsilon_k} \cdots A_{\varepsilon_1} v\|^p < m^{(-n/s+1/p-t)kp}.$$

Clearly, for this k, there is an $\eta > 0$ satisfying that for any $b \in \ell(\Omega)$ with $\|a-b\|_1 < \eta$, we have

$$\sum_{\varepsilon_1,\ldots,\varepsilon_k \in E} \|A_{\varepsilon_k} \cdots A_{\varepsilon_1} - B_{\varepsilon_k} \cdots B_{\varepsilon_1}\|^p < m^{(-n/s+1/p-t)kp}.$$

Note that $V_n \cap \ell(K)$ is an invariant subspace of any A_ε and B_ε, $\varepsilon \in E$. Consequently,

$$\max_{\substack{v \in V_n \cap \ell(K) \\ \|v\|=1}} \sum_{\varepsilon_1,\ldots,\varepsilon_k \in E} \|(A_{\varepsilon_k} \cdots A_{\varepsilon_1})v - (B_{\varepsilon_k} \cdots B_{\varepsilon_1})v\|^p < m^{(-n/s+1/p-t)kp}.$$

It follows from the triangle inequality that

$$\max_{\substack{v \in V_n \cap \ell(K) \\ \|v\|=1}} \sum_{\varepsilon_1,\ldots,\varepsilon_k \in E} \|B_{\varepsilon_k} \cdots B_{\varepsilon_1} v\|^p < m^{(-n/s+1/p-t_1)kp}, \tag{4.1}$$

where the positive number t_1 is defined by $2m^{(-n/s+1/p-t)kp} = m^{(-n/s+1/p-t_1)kp}$. Equality (2.2) tells us now

$$\rho_p(\{B_\varepsilon|_{V_n \cap \ell(K)} : \varepsilon \in E\}) \leqslant m^{(-n/s+1/p-t_1)} < m^{-n/s+1/p},$$

and the assertion follows from Result 2.1. ∎

So, in fact, the convergence of $(Q_b^k\phi)_{k\geqslant 0}$ follows readily from the continuity of the joint spectral radius ρ_p.

Our goal is now to estimate the perturbation of the limit function in terms of the perturbation of the mask, i.e., we want to show that

$$\|Q_a^k\phi_0 - Q_b^k\phi_0\|_{W_p^n(\mathbb{R}^s)} \leqslant c\|a-b\|_1, \qquad k=1,2,\ldots,$$

where a, b meet the assumptions of Theorem 4.1. We use the initial function ϕ_0 defined in (3.4). Then, choosing $g = Q_a^k\phi_{0,n} - Q_b^k\phi_{0,n}$, the second inequality in (3.5) implies that

$$\sum_{|\mu|=n} \|D^\mu Q_a^k\phi_{0,n} - D^\mu Q_b^k\phi_{0,n}\|_p \leqslant c m^{(n/s-1/p)k} \sum_{|\mu|=n} \|\Delta^\mu a_k - \Delta^\mu b_k\|_p.$$

Hence, we have to estimate the norm $\|\Delta^\mu a_k - \Delta^\mu b_k\|_p$ for $|\mu| = n$.

In order to obtain this estimate we first need

LEMMA 4.2. *Assume that the masks $a, b \in \ell_0(\mathbb{Z}^s)$ satisfy* (1.3) *and the sum rules of order $n+1$ in* (2.9). *Then for any $v \in V_{n-1}$, we have*

$$(B_\varepsilon - A_\varepsilon)v \in V_n \qquad \forall \varepsilon \in E.$$

Proof. We claim that for any a satisfying sum rules of order $n+1$ and any $p \in \Pi_n$, there is a polynomial $q \in \Pi_{n-1}$ such that

$$\begin{aligned}&\sum_{\alpha\in\mathbb{Z}^s} p(-\alpha)a(\varepsilon + M\alpha - \beta)\\ &\quad = p(M^{-1}(\varepsilon-\beta)) + q(\varepsilon-\beta) \qquad \forall \varepsilon \in E \quad \text{and} \quad \forall \beta \in \mathbb{Z}^s. \qquad (4.2)\end{aligned}$$

In fact, it follows from Taylor's formula that

$$p(-\alpha) = \sum_{|\mu|\leqslant n} \frac{D^\mu p(M^{-1}(\varepsilon-\beta))}{\mu!} (-M^{-1}(M\alpha - \beta + \varepsilon))^\mu.$$

Therefore,

$$\begin{aligned}&\sum_{\alpha\in\mathbb{Z}^s} p(-\alpha)a(\varepsilon + M\alpha - \beta)\\ &\quad = \sum_{|\mu|\leqslant n} \frac{D^\mu p(M^{-1}(\varepsilon-\beta))}{\mu!} \sum_{\alpha\in\mathbb{Z}^s} (-M^{-1}(M\alpha-\beta+\varepsilon))^\mu a(\varepsilon + M\alpha - \beta).\end{aligned}$$

Note that a satisfies (1.3) and (2.9), i.e., we have $\sum_{\alpha\in\mathbb{Z}^s} a(\varepsilon + M\alpha - \beta) = 1$ and

$$\sum_{\alpha\in\mathbb{Z}^s} (-M^{-1}(M\alpha - \beta + \varepsilon))^\mu a(\varepsilon + M\alpha - \beta) = \sum_{\alpha\in\mathbb{Z}^s} (-\alpha)^\mu a(M\alpha), \qquad |\mu| \leqslant n,$$

for all $\varepsilon \in E$ and $\beta \in \mathbb{Z}^s$. Hence, we obtain

$$\sum_{\alpha \in \mathbb{Z}^s} p(-\alpha) a(\varepsilon + M\alpha - \beta)$$

$$= p(M^{-1}(\varepsilon - \beta)) + \sum_{0 < |\mu| \leqslant n} \sum_{\alpha \in \mathbb{Z}^s} \frac{D^\mu p(M^{-1}(\varepsilon - \beta))}{\mu!} (-\alpha)^\mu a(M\alpha).$$

This proves (4.2). Using (4.2) for b instead of a we get a polynomial $g \in \Pi_{n-1}$ such that

$$\sum_{\alpha \in \mathbb{Z}^s} p(-\alpha)(b(\varepsilon + M\alpha - \beta) - a(\varepsilon + M\alpha - \beta)) = g(\varepsilon - \beta) \qquad \forall \varepsilon \in E \quad \text{and} \quad \forall \beta \in \mathbb{Z}^s.$$

For any $v \in V_{n-1}$ and for any $p \in \Pi_n$, it follows by (2.4) that

$$\sum_{\alpha \in \mathbb{Z}^s} p(-\alpha)(B_\varepsilon - A_\varepsilon) v(\alpha) = \sum_{\beta \in \mathbb{Z}^s} g(\varepsilon - \beta) v(\beta) = 0 \qquad \forall \varepsilon \in E.$$

The proof is complete. ∎

LEMMA 4.3. *Suppose that* $\Omega \subseteq \mathbb{Z}^s$ *is a finite set and that the cascade algorithm corresponding to* $a \in \ell(\Omega)$ *converges for every* $\phi \in W_n$ *in* $W_p^n(\mathbb{R}^s)$*-norm. Further, let* $b \in \ell(\Omega)$ *satisfy* (1.3), *the sum rules of order* $n+1$ *and* $\|a - b\|_1 < \eta$, *where* η *is chosen such that the assertion of Theorem* 4.1 *holds. Then, there is a positive number* c *such that we have*

$$\|\Delta^\mu a_k - \Delta^\mu b_k\|_p \leqslant c \|a - b\|_1 m^{(-n/s + 1/p)k} \qquad \forall |\mu| = n \quad \textit{and} \ k = 1, 2, \ldots,$$

where c *is independent of* b *and* k.

Proof. Let K be given in (2.8). By (2.6) and the equality

$$B_{\varepsilon_k} \cdots B_{\varepsilon_1} - A_{\varepsilon_k} \cdots A_{\varepsilon_1} = \sum_{j=1}^{k} B_{\varepsilon_k} \cdots B_{\varepsilon_{j+1}} (B_{\varepsilon_j} - A_{\varepsilon_j}) A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_1},$$

we obtain

$$\|(b_k - a_k) * v\|_p$$

$$= \left(\sum_{\varepsilon_1,\ldots,\varepsilon_k \in E} \sum_{\gamma \in K} |B_{\varepsilon_k} \cdots B_{\varepsilon_1} v(\gamma) - A_{\varepsilon_k} \cdots A_{\varepsilon_1} v(\gamma)|^p \right)^{1/p}$$

$$\leqslant \sum_{j=1}^{k} \left(\sum_{\varepsilon_1,\ldots,\varepsilon_k \in E} \sum_{\gamma \in K} |B_{\varepsilon_k} \cdots B_{\varepsilon_{j+1}} (B_{\varepsilon_j} - A_{\varepsilon_j}) A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_1} v(\gamma)|^p \right)^{1/p},$$

where we have used that by Lemma 2.2 there is some integer $k_0 > 0$ such that both $A_{\varepsilon_k} \ldots A_{\varepsilon_1} v$ and $B_{\varepsilon_k} \ldots B_{\varepsilon_1} v$ are in $\ell(K)$ for all $k \geqslant k_0$.

Thus,

$$\|(b_k - a_k) * v\|_p \leqslant \sum_{j=1}^{k} \left(\sum_{\varepsilon_1,\ldots,\varepsilon_k \in E} \|B_{\varepsilon_k} \cdots B_{\varepsilon_{j+1}} (B_{\varepsilon_j} - A_{\varepsilon_j}) A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_1} v\|_p^p \right)^{1/p}. \tag{4.3}$$

Let $|\mu| = n$. Note that $\|\Delta^\mu a_{j-1}\|_p^p = \sum_{\varepsilon_1,\ldots,\varepsilon_{j-1} \in E} \|A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_1} \Delta^\mu \delta\|_p^p$. Hence, by (3.6) in Theorem 3.2, there is a constant $c_1 > 0$ such that for any j

$$\sum_{\varepsilon_1,\ldots,\varepsilon_{j-1} \in E} \|A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_1} \Delta^\mu \delta\|_p^p \leqslant c_1 m^{(-n/s+1/p)(j-1)p}. \tag{4.4}$$

On the other hand, from $\Delta^\mu \delta \in V_{n-1}$ and Lemma 4.2 it follows that

$$(B_{\varepsilon_j} - A_{\varepsilon_j}) A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_1} \Delta^\mu \delta \in V_n \qquad \forall \varepsilon_1, \ldots, \varepsilon_j \in E.$$

Moreover, by Theorem 4.1 we already know that the cascade algorithm corresponding to b converges in $W_p^n(\mathbb{R}^s)$-norm and by (4.1) there are a positive number t_1 and a constant c_2 such that

$$\sum_{\varepsilon_{j+1},\ldots,\varepsilon_k \in E} \|B_{\varepsilon_k} \cdots B_{\varepsilon_{j+1}} (B_{\varepsilon_j} - A_{\varepsilon_j}) A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_1} \Delta^\mu \delta\|_p^p$$

$$\leqslant c_2 m^{(-n/s+1/p-t_1)(k-j)p} \|(B_{\varepsilon_j} - A_{\varepsilon_j}) A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_1} \Delta^\mu \delta\|_p^p \qquad \forall k > j.$$

This together with (4.4) implies by $\|B_{\varepsilon_j} - A_{\varepsilon_j}\|_p^p \leqslant \|a - b\|_1^p$

$$\sum_{\varepsilon_1,\ldots,\varepsilon_{j-1} \in E} \sum_{\varepsilon_{j+1},\ldots,\varepsilon_k \in E} \|B_{\varepsilon_k} \cdots B_{\varepsilon_{j+1}} (B_{\varepsilon_j} - A_{\varepsilon_j}) A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_1} \Delta^\mu \delta\|_p^p$$

$$\leqslant c_3 m^{(-n/s+1/p-t_1)(k-j)p} m^{(-n/s+1/p)(j-1)p} \|a - b\|_1^p,$$

where c_3 is some constant which is independent of b and k. It follows from (4.3) that

$$\|(b_k - a_k) * \Delta^{\mu}\delta\|_p \leqslant c_3^{1/p} \|a-b\|_1 m^{(-n/s+1/p)(k-1)} \sum_{j=1}^{k} m^{-(k-j)t_1}, \qquad k = 1, 2, \ldots .$$

Hence, the assertion follows. ∎

We are now ready to present the main theorem of this section.

THEOREM 4.4. *Let Ω be a finite set in $\mathbb{Z}^s$. Assume that the cascade algorithm corresponding to $a \in \ell(\Omega)$ converges for every $\phi \in W_n$ in $W_p^n(\mathbb{R}^s)$-norm. Then, there exists a positive constant η such that, for any $b \in \ell(\Omega)$ satisfying* (1.3) *and the sum rules of order $n+1$ with $\|a-b\|_1 < \eta$, the cascade algorithm corresponding to b converges for every $\phi \in W_n$ in $W_p^n(\mathbb{R}^s)$-norm. Moreover, there exists a constant c, which is independent of b and k, such that*

$$\|Q_a^k \phi_{0,n} - Q_b^k \phi_{0,n}\|_{W_p^n(\mathbb{R}^s)} \leqslant c\|a-b\|_1, \qquad k = 1, 2, \ldots, \tag{4.5}$$

where $\phi_{0,n}$ is given in (3.4). *Consequently, we find for the limit functions*

$$\|\phi_a - \phi_b\|_{W_p^n(\mathbb{R}^s)} \leqslant c\|a-b\|_1. \tag{4.6}$$

Proof. By Theorem 4.1, we know that for $b \in \ell(\Omega)$ satisfying the sum rules of order $n+1$ and with $\|a-b\| < \eta$ for some suitable η the cascade algorithm corresponding to mask b converges for every $\phi \in W_n$ in $W_p^n(\mathbb{R}^s)$-norm. Therefore, $\phi_b \in W_p^n(\mathbb{R}^s)$. Since $\phi_{0,n} \in W_n$, the cascade algorithm converges for $\phi_{0,n}$ for a and b, i.e., we have

$$\lim_{k\to\infty} \|Q_a^k \phi_{0,n} - \phi_a\|_{W_p^n(\mathbb{R}^s)} = \lim_{k\to\infty} \|Q_b^k \phi_{0,n} - \phi_b\|_{W_p^n(\mathbb{R}^s)} = 0.$$

Inequality (4.6) follows now from (4.5).

In order to prove (4.5), we appeal to Theorem 3.2. Put $\lambda = \Delta^{\mu} a_k - \Delta^{\mu} b_k$ in (3.5). This corresponds to $g = Q_a^k \phi_{0,n} - Q_b^k \phi_{0,n}$. Then the second inequality in (3.5) yields for some constant c_1 and for $k = 1, 2, \ldots$

$$\sum_{|\mu|=n} \|D^{\mu}(Q_a^k \phi_{0,n} - Q_b^k \phi_{0,n})\|_p \leqslant c_1 m^{(n/s-1/p)k} \sum_{|\mu|=n} \|\Delta^{\mu} a_k - \Delta^{\mu} b_k\|_p.$$

Together with Lemma 4.3, it in turn implies

$$\sum_{|\mu|=n} \|D^{\mu}(Q_a^k \phi_{0,n} - Q_b^k \phi_{0,n})\|_p \leqslant c_2 \|a-b\|_1, \qquad k = 1, 2, \ldots, \tag{4.7}$$

where c_2 is some positive number being independent of b and k.

As shown in Corollary 3.4, the cascade algorithm corresponding to a also converges for every $\phi \in W_{n'}$ in $W_p^{n'}(\mathbb{R}^s)$-norm with $n' < n$. Replacing n with n' in (4.7) and then taking the sum of the resulting inequalities we obtain (4.5). ∎

We obtain the following corollary.

COROLLARY 4.5. *Let Ω be a finite set in $\mathbb{Z}^s$. Suppose that ϕ_a is a refinable function in $W_p^n(\mathbb{R}^s)$ corresponding to mask $a \in \ell(\Omega)$ and the shifts of ϕ_a are stable. Then there are positive constants η and c such that, for any $b \in \ell(\Omega)$ satisfying* (1.3), *the sum rules of order $n+1$ and $\|a-b\|_1 < \eta$, the refinable distribution ϕ_b is in $W_p^n(\mathbb{R}^s)$ and satisfies* (4.6).

Proof. By the stability of the shifts of ϕ_a, the cascade algorithm corresponding to a converges on W_n in $W_p^n(\mathbb{R}^s)$-norm. This conclusion has been established in [19] for $p = 2$. The method works for general $p \geqslant 1$. Now, using Theorem 3.2, the proof is analogous to that of Theorem 4.4. ∎

Remark. The proof of estimate (4.5) is strongly based on the second inequality in (3.5). This inequality in turn has been shown for our initial function ϕ_0 in (3.4) using relation (3.2). Since not every function f in W_n satisfies the relation $D^\mu f = \Delta^\mu g$ for some suitable g as in (3.2), the arguments in this paper fail to work for a general initial function in W_n. This difficulty has been overcome recently by Han [9]. In this paper, he established inequality (4.5) for any initial function in W_n.

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